Supplemental Material to article "Polynomial search and global modeling – two new algorithms for modeling chaos" by Mangiarotti S., Coudret R., Drapeau L. & Jarlan L., 2012, *Physical Review E*.

Global models' formulation and associated transition matrices for the Rössler system, the electrodissolution of copper in phosphoric acid and the cycles of rainfed wheat observed from satellite in North Morocco.

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Two new algorithms *Polynomial Model Search* and *Global Modelling* were introduced in the main body of the present work [1]. Three systems are considered to test the robustness and the accuracy of both algorithms: one theoretical, one experimental and one environmental. The Rössler sytem [2] is used as theoretical benchmark. The three variables of this system were considered one by one in order to account for the different degrees of observability [3]. Measurements resulting from the electrodissolution of copper in phosphoric acid [4] were used as an experimental case. The cycle of rainfed wheat was chosen as an environmental case of study, based on vegetation index measured from satellite [5].

Global modeling aims at building mathematical models of concise description. Several models were obtained for each of the three systems considered in the study. These models are 3-dimensionnal (n = 3) and rely on Equation (3) presented in [1], in a polynomial formulation. The object of the present Supplemental Material is to provide explicitly the formulation of the models obtained for each of these systems and the transition matrices used for their validation (or invalidation). The transition matrices of the models are provided only when a partition could be obtained from the first return map. Transition matrices estimated from the original data set are also provided for comparison. The detailed introduction as well as a complete description of the data sets, theoretical background and algorithms are provided in [1] together with a discussion of the results.

1 The Rössler system

1.1 Rössler-x_2

Two models were obtained from the Rössler-x_2 variable, the 9-term model (of maximum degree q = 2) reads:

$$
\begin{aligned}
P(X_1, X_2, X_3) = & -4.571\,956.10^{2} - 2.076\,100\,.10^{1} X_3 \\
& + 3.855\,515.10^{1} X_2 + 9.570\,891.10^{-1} X_2 X_3 - 3.127\,505\, X_2^2 \\
& - 9.350\,558.10^{2} X_1 - 3.099\,332\, X_1 X_3 + 4.760\,105\,.10^{1} X_1 X_2 \\
& - 1.217\,603.10^{2} X_1^2
\end{aligned}
\qquad (1)
$$

and the 7-term model (of maximum degree q = 2, also) reads:

$$P\left(X_1, X_2, X_3\right) = -1.3824700\,.10^{1} X_3 + 6.927122.10^{-1} X_2 X_3 - 3.3915271\, X_2^2 - 6.970505.10^{2} X_1 - 1.900433\, X_1 X_3 + 3.439617\,.10^{1} X_1 X_2 - 9.389699\,.10^{1} X_1^2 \quad (2)$$

The (Markov) transfer matrix and corresponding binary transfer matrix were estimated from the first return maps presented in Figure 5 (see Tables 2 & 3 for details). For the 9-term model, these matrices read:

$$\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.28 & 0.23 & 0.23 & 0.27 \end{pmatrix} \end{array} \qquad \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0.07 & 0.05 & 0.06 & 0.07 \\ 0.07 & 0.05 & 0.05 & 0.06 \\ 0.07 & 0.06 & 0.06 & 0.07 \\ 0.07 & 0.06 & 0.06 & 0.07 \end{pmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array} \quad (3)$$

and for the 7-term model:

$$\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.14 & 0.36 & 0.36 & 0.14 \end{pmatrix} \end{array} \qquad \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0.16 & 0.06 & 0.07 & 0.01 \\ 0.07 & 0.11 & 0.12 & 0.06 \\ 0.07 & 0.11 & 0.12 & 0.06 \\ 0.01 & 0.05 & 0.06 & \mathbf{0} \end{pmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \mathbf{0} \end{pmatrix} \end{array} \quad (4)$$

These matrices should be compared to the following ones estimated from the original signal of variable x_2:

$$\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.23 & 0.28 & 0.25 & 0.25 \end{pmatrix} \end{array} \qquad \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0.02 & 0.10 & 0.06 & 0.08 \\ 0.07 & 0.07 & 0.04 & 0.07 \\ 0.05 & 0.05 & 0.09 & 0.06 \\ 0.05 & 0.07 & 0.06 & 0.05 \end{pmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array} \quad (5)$$

1.2 Rössler-x_1

Three models were obtained from the x_1 variable. The 13-term model (of maximum degree q = 4) reads:

$$
\begin{aligned}
P(X_1, X_2, X_3) = & -1.218\,738.10^{1}\ X_3 - 7.764\,914.10^{-3}\, X_3^2 \\
& - 6.308\,658.10^{-5}\ X_2 X_3^2 - 5.553\,971.10^{2}\ X_1 - 2.849\,993\, X_1 X_3 \\
& + 7.498\,988.10^{-3}\ X_1 X_3^2 - 0.267\,307\, X_1 X_2 X_3 + 1.102\,677\, X_1 X_2^2 \\
& - 1.964\,489.10^{-3}\ X_1 X_2^2 X_3 + 3.613\,635.10^{-1}\, X_1^2 X_3 \\
& - 1.412\,911.10^{1}\, X_1^2 X_2 + 1.931\,847.10^{-1}\, X_1^3 X_3 - 1.572\,845\, X_1^3 X_2
\end{aligned}
\qquad (6)
$$

and its tuned version was obtained by multiplying by 0.9 the parameter corresponding to term X_3. The 9-term model (of maximum degree q = 4, also) reads:

$$
\begin{aligned}
P(X_1, X_2, X_3) = & -1.249\,580.10^{1}\, X_3 - 7.266\,211.10^{2}\, X_1 \\
& - 3.141\,190 X_1 X_3 + 4.819\,542.10^{-3}\, X_1 X_3^2 - 1.933\,067.10^{-1}\, X_1 X_2 X_3 \\
& + 1.415\,384 X_1 X_2^2 - 1.259\,086.10^{1}\, X_1^2 X_2 + 2.257\,435.10^{-1}\, X_1^3\ X_3 \\
& - 1.294\,766\ X_1^3 X_2
\end{aligned}
\qquad (7)
$$

and the 10-term model (maximum degree q = 3) reads:

$$
\begin{aligned}
P(X_1, X_2, X_3) = & -1.671440.\,10^{1}\ X_3 - 2.847528.10^{-1}\ X_2 X_3 + 1.481552\ X_2^2 \\
& - 9.143745.10^{2}\ X_1 + 2.101548.10^{-3}\ X_1 X_3^2 - 3.109967.10^{1}\, X_1 X_2 \\
& + 7.998291.10^{-1}\, X_1 X_2^2 + 1.237751\ X_1^2 X_3 - 7.199582\ X_1^2 X_2 \\
& + 5.430887.10^{1}\, X_1^3
\end{aligned}
\qquad (8)
$$

The tuned model derived from this latter 10-term model was obtained by multiplying by 0.92 the parameter corresponding to term X_3. The following (Markov) transfer matrix and corresponding binary transfer matrix were estimated from the first return maps shown in Figure 6 (see Tables 2 & 3 for details). For the 13-term model, these matrices read:

$$
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{pmatrix} 0.21 & 0.35 & 0.44 & \mathbf{0} \end{pmatrix} \\
\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \mathbf{0} & 0.10 & 0.12 & \mathbf{0} \\ 1 & 0.10 & 0.11 & 0.14 & \mathbf{0} \\ 2 & 0.10 & 0.13 & 0.19 & \mathbf{0} \\ 3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}
\end{array}
\qquad
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{pmatrix} 1 & 1 & 1 & \mathbf{0} \end{pmatrix} \\
\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \mathbf{0} & 1 & 1 & \mathbf{0} \\ 1 & 1 & 1 & 1 & \mathbf{0} \\ 2 & 1 & 1 & 1 & \mathbf{0} \\ 3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}
\end{array}
\qquad (9)
$$

and after tuning the parameter corresponding to term X_3:

$$
\begin{array}{cccc} 0 & 1 & 2 & 3 \\ \end{array}
\qquad
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array}
$$

$$
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{pmatrix} 0.08 & 0.39 & 0.38 & 0.16 \end{pmatrix} \\
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c|c} & \\ \end{array}
\end{array}
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \\
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array}
\end{array}
$$

$$
\begin{array}{c}
0 \\ 1 \\ 2 \\ 3
\end{array}
\begin{pmatrix} \mathbf{0} & 0.05 & 0.03 & 0.05 \\ 0.05 & 0.10 & 0.08 & 0.10 \\ 0.03 & 0.07 & 0.06 & 0.07 \\ 0.05 & 0.10 & 0.08 & 0.10 \end{pmatrix}
\qquad
\begin{array}{c}
0 \\ 1 \\ 2 \\ 3
\end{array}
\begin{pmatrix} \mathbf{0} & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}
\tag{10}
$$

For the 9-term model, transfer matrices read:

$$
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{pmatrix} \mathbf{0} & 0.35 & 0.60 & 0.05 \end{pmatrix}
\end{array}
\qquad
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{pmatrix} \mathbf{0} & 1 & 1 & 1 \end{pmatrix}
\end{array}
$$

$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
1 & \mathbf{0} & 0.07 & 0.17 & 0.04 \\
2 & \mathbf{0} & 0.17 & 0.26 & 0.13 \\
3 & \mathbf{0} & 0.03 & 0.13 & \mathbf{0}
\end{array}
\qquad
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
1 & \mathbf{0} & 1 & 1 & 1 \\
2 & \mathbf{0} & 1 & 1 & 1 \\
3 & \mathbf{0} & 1 & 1 & \mathbf{0}
\end{array}
\tag{11}
$$

For the 10-term model, transfer matrices read:

$$
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{pmatrix} 0.15 & 0.40 & 0.45 & \mathbf{0} \end{pmatrix}
\end{array}
\qquad
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{pmatrix} 1 & 1 & 1 & \mathbf{0} \end{pmatrix}
\end{array}
$$

$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & \mathbf{0} & 0.04 & 0.13 & \mathbf{0} \\
1 & 0.03 & 0.14 & 0.24 & \mathbf{0} \\
2 & 0.10 & 0.21 & 0.11 & \mathbf{0} \\
3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}
\qquad
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & \mathbf{0} & 1 & 1 & \mathbf{0} \\
1 & 1 & 1 & 1 & \mathbf{0} \\
2 & 1 & 1 & 1 & \mathbf{0} \\
3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}
\tag{12}
$$

and after tuning the coefficient corresponding to term X_3:

$$
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{pmatrix} 0.26 & 0.29 & 0.20 & 0.25 \end{pmatrix}
\end{array}
\qquad
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}
\end{array}
$$

$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & 0.04 & 0.07 & 0.06 & 0.10 \\
1 & 0.09 & 0.08 & 0.06 & 0.10 \\
2 & 0.06 & 0.04 & 0.04 & 0.06 \\
3 & 0.07 & 0.06 & 0.04 & 0.03
\end{array}
\qquad
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1
\end{array}
\tag{13}
$$

These matrices should be compared to the matrices estimated from the original signal of variable x_1:

$$
\begin{array}{cccc} 0 & 1 & 2 & 3 \\ \end{array}
\qquad
\begin{array}{cccc} 0 & 1 & 2 & 3 \\ \end{array}
$$

$$
\begin{pmatrix} 0.22 & 0.29 & 0.23 & 0.25 \end{pmatrix}
\qquad
\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}
$$

$$
\begin{array}{c|c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0.02 & 0.07 & 0.06 & 0.07 \\ 0.10 & 0.07 & 0.04 & 0.09 \\ 0.04 & 0.05 & 0.08 & 0.06 \\ 0.06 & 0.07 & 0.05 & 0.06 \end{pmatrix}
\end{array}
\qquad
\begin{array}{c|c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}
\end{array}
\qquad (14)
$$

The small differences, when comparing the matrices obtained from variable x_2 (with Eq. 15) with these latter ones obtained from variable x_1 (Eq. 24), may arise from the imprecision when choosing the boundary limit between two symbols in the partition of the first return map.

1.3 Rössler-x_3

One 30-term model was obtained from the x_3 variable of the Rössler system. The model (of maximum degree q = 5) reads:

$$
\begin{aligned}
P(X_1, X_2, X_3) = & \, 1.469\,183\,837\; X_2 X_3 + 3.298\,435\,897 \cdot 10^{1}\; X_2^2 \\
& - 5.079\,722\,432 \cdot 10^{-6}\; X_2^2\, X_3^2 - 0.095\,501\,855\; X_2^3 \\
& - 2.454\,817\,807 \cdot 10^{-5}\; X_2^3\, X_3 - 0.000\,743\,108\; X_2^4 - 8.516\,901\,345\; X_1\, X_3 \\
& - 0.007\,078\,151\; X_1\, X_3^2 + 3.822\,423\,364 \cdot 10^{-6}\; X_1\, X_3^3 \\
& - 0.457\,986\,519\; X_1\, X_2\, X_3 + 3.536\,469\,082 \cdot 10^{-5}\; X_1\, X_2\, X_3^2 \\
& - 1.222\,762\,045 \cdot 10^{1}\; X_1\, X_2^2 + 4.453\,556\,554 \cdot 10^{-7}\; X_1\, X_2^2\, X_3^2 \\
& + 0.010\,954\,386\; X_1\, X_2^3 + 0.003\,155\,023\; X_1^2\, X_3^2 \\
& - 3.343\,660\,288 \cdot 10^{-7}\; X_1^2\, X_3^3 - 3.328\,680\,543 \cdot 10^{1}\; X_1^2\, X_2 \\
& + 0.065\,324\,718\; X_1^2\, X_2\, X_3 - 1.635\,579\,935 \cdot 10^{-6}\; X_1^2\, X_2\, X_3^2 \\
& + 1.294\,265\,505\; X_1^2\, X_2^2 + 0.000\,160\,434\; X_1^2\, X_2^2\, X_3 \\
& - 0.001\,047\,247\; \mathrm{X}_1^2\, X_2^3 - 6.374\,848\,605 \cdot 10^{2}\; X_1^3 + 0.876\,100\,551\; X_1^3\, X_3 \\
& - 0.000\,320\,012\; X_1^3\, X_3^2 + 4.724\,411\,452\; X_1^3\, X_2 \\
& - 0.001\,967\,624\; X_1^3\, X_2\, X_3 + 2.265\,117\,133 \cdot 10^{2}\; X_1^4 \\
& - 0.122\,449\,442\; \mathrm{X}_1^4\, X_3 - 2.165\,857\,333 \cdot 10^{1}\; X_1^5
\end{aligned}
\qquad (15)
$$

The tuned version of the model was obtained by multiplying by 1.002 the parameter corresponding to term X_2^2. The following (Markov) transfer matrix and corresponding binary transfer matrix were estimated from the first return maps shown in Figure 7 (see Tables 2 & 3 for details). For the 30-term (not tuned) model, these matrices read:

$$\begin{array}{cc} & \begin{array}{cccc}0 & 1 & 2 & 3\end{array}\\ & \begin{pmatrix}0.21 & 0.59 & 0.20 & \mathbf{0}\end{pmatrix}\\ & \begin{array}{cccc}0 & 1 & 2 & 3\end{array}\\ \begin{array}{c}0\\1\\2\\3\end{array} & \begin{pmatrix}0.02 & 0.16 & 0.01 & \mathbf{0}\\ 0.17 & 0.31 & 0.16 & \mathbf{0}\\ 0.01 & 0.15 & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix}\end{array} \qquad \begin{array}{cc} & \begin{array}{cccc}0 & 1 & 2 & 3\end{array}\\ & \begin{pmatrix}1 & 1 & 1 & \mathbf{0}\end{pmatrix}\\ & \begin{array}{cccc}0 & 1 & 2 & 3\end{array}\\ \begin{array}{c}0\\1\\2\\3\end{array} & \begin{pmatrix}1 & 1 & 1 & \mathbf{0}\\ 1 & 1 & 1 & \mathbf{0}\\ 1 & 1 & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix}\end{array} \tag{16}$$

and after tuning the coefficient of term X_2^2, transfer matrices become:

$$\begin{array}{cc} & \begin{array}{cccc}0 & 1 & 2 & 3\end{array}\\ & \begin{pmatrix}0.16 & 0.39 & 0.17 & 0.28\end{pmatrix}\\ & \begin{array}{cccc}0 & 1 & 2 & 3\end{array}\\ \begin{array}{c}0\\1\\2\\3\end{array} & \begin{pmatrix}0.02 & 0.08 & 0.02 & 0.05\\ 0.08 & 0.15 & 0.07 & 0.09\\ 0.03 & 0.05 & 0.03 & 0.06\\ 0.04 & 0.10 & 0.05 & 0.08\end{pmatrix}\end{array} \qquad \begin{array}{cc} & \begin{array}{cccc}0 & 1 & 2 & 3\end{array}\\ & \begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}\\ & \begin{array}{cccc}0 & 1 & 2 & 3\end{array}\\ \begin{array}{c}0\\1\\2\\3\end{array} & \begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{pmatrix}\end{array} \tag{17}$$

These distributions and matrices should be compared to the transfer matrices estimated from the original variable x_3:

$$\begin{array}{cc} & \begin{array}{cccc}0 & 1 & 2 & 3\end{array}\\ & \begin{pmatrix}0.23 & 0.28 & 0.24 & 0.25\end{pmatrix}\\ & \begin{array}{cccc}0 & 1 & 2 & 3\end{array}\\ \begin{array}{c}0\\1\\2\\3\end{array} & \begin{pmatrix}0.03 & 0.07 & 0.06 & 0.07\\ 0.10 & 0.07 & 0.04 & 0.08\\ 0.04 & 0.05 & 0.09 & 0.06\\ 0.06 & 0.07 & 0.06 & 0.05\end{pmatrix}\end{array} \qquad \begin{array}{cc} & \begin{array}{cccc}0 & 1 & 2 & 3\end{array}\\ & \begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}\\ & \begin{array}{cccc}0 & 1 & 2 & 3\end{array}\\ \begin{array}{c}0\\1\\2\\3\end{array} & \begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{pmatrix}\end{array} \tag{18}$$

Here again, small differences (when comparing to the matrices resulting from variables x_2 and x_1, see Eq. 15 & 24) may arise from the imprecision when choosing the boundary limits for the symbols' partition in the first return map.

2 The electrodissolution of copper in phosphoric acid

Two parameterizations of the 21-term model (of maximum degree $q = 4$) were obtained for the electrodissolution of copper in phosphoric acid. The first one (noted 21-p., see figures 8c and 8d) reads:

$$
\begin{aligned}
P(X_1, X_2, X_3) = {} & 4.376\,214.10^{3} + 2.540\,685.10^{-1}\,X_3^2 - 3.502\,923.10^{-5}\,X_3^3 \\
& + 1.088\,926.10^{-9}\,X_3^4 + 6.370\,651.10^{-1}\,X_2 X_3 - 4.782\,412.10^{-4}\,X_2 X_3^2 \\
& + 1.522\,106.10^{-8}\,X_2 X_3^3 + 8.113\,909\,X_2^2 - 1.198\,235.10^{-2}\,X_1 X_3^2 \\
& + 8.081\,680.10^{-7}\,X_1 X_3^3 + 1.541\,398.10^{1}\,X_1 X_2 + 9.296\,108.10^{-6}\,X_1 X_2 X_3^2 \\
& - 6.117\,464.10^{-5}\,X_1 X_2^2 X_3 - 5.280\,080.10^{-5}\,X_1 X_2^3 - 5.282\,890.10^{-3}\,X_1^2 X_3 \\
& + 1.415\,405.10^{-4}\,X_1^2 X_3^2 - 1.762\,102.10^{-1}\,X_1^2 X_2 - 1.343\,480.10^{-4}\,X_1^2 X_2 X_3 \\
& - 4.303\,322.10^{-3}\,X_1^2 X_2^2 - 5.855\,790.10^{-2}\,X_1^3 - 4.677\,767.10^{-3}\,X_1^3 X_2
\end{aligned} \tag{19}
$$

and the second one (noted 21-p.*, see Figure 8e and 8f) reads:

$$
\begin{aligned}
P(X_1, X_2, X_3) = {} & 3.487\,242 \cdot 10^{3} + 3.556\,419 \cdot 10^{-1}\,X_3^2 - 5.142\,267 \cdot 10^{-5}\,X_3^3 \\
& + 1.817\,063 \cdot 10^{-9}\,X_3^4 + 5.612\,236 \cdot 10^{-1}\,X_2 X_3 - 4.135\,642 \cdot 10^{-4}\,X_2 X_3^2 \\
& + 1.349\,334 \cdot 10^{-8}\,X_2 X_3^3 + 6.591\,543\,X_2^2 - 1.628\,945 \cdot 10^{-2}\,X_1 X_3^2 \\
& + 1.171\,202 \cdot 10^{-6}\,X_1 X_3^3 + 6.503\,816\,X_1 X_2 + 7.907\,225 \cdot 10^{-6}\,X_1 X_2 X_3^2 \\
& - 5.248\,275 \cdot 10^{-5}\,X_1 X_2^2 X_3 - 4.267\,778 \cdot 10^{-5}\,X_1 X_2^3 - 4.468\,080 \cdot 10^{-3}\,X_1^2 X_3 \\
& + 1.878\,962 \cdot 10^{-4}\,X_1^2 X_3^2 - 1.260\,730 \cdot 10^{-1}\,X_1^2 X_2 - 1.183\,509 \cdot 10^{-4}\,X_1^2 X_2 X_3 \\
& - 3.671\,600 \cdot 10^{-3}\,X_1^2 X_2^2 - 4.723\,065 \cdot 10^{-2}\,X_1^3 - 6.952\,164 \cdot 10^{-3}\,X_1^3 X_2
\end{aligned} \tag{20}
$$

The tuned version of this second model (noted 21-p.$^{*\,\mathrm{opt}}$) was obtained by multiplying by 1.002 the parameter corresponding to term $X_1^3 X_2$ (see Figure 8g). The (Markov) transfer matrix and corresponding binary transfer matrix were estimated from the first return map shown in Figure 8, (see Tables 2 & 3 for details). For the 21-term model (21-p.) , these matrices read:

$$
\begin{array}{cc}
 & \begin{array}{cc} 0 & 1 \end{array} \\
 & \begin{pmatrix} 0.30 & 0.70 \end{pmatrix} \\
 & \begin{array}{cc} 0 & 1 \end{array} \\
\begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} \mathbf{0} & 0.25 \\ 0.25 & 0.5 \end{pmatrix}
\end{array}
\qquad\qquad
\begin{array}{cc}
 & \begin{array}{cc} 0 & 1 \end{array} \\
 & \begin{pmatrix} 1 & 1 \end{pmatrix} \\
 & \begin{array}{cc} 0 & 1 \end{array} \\
\begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} \mathbf{0} & 1 \\ 1 & 1 \end{pmatrix}
\end{array} \tag{21}
$$

And for the alternative 21-term model, the following matrices were obtained using the tuned version of the model (21-p.$^{*\,\mathrm{opt}}$):

$$
\begin{array}{cc}
 & \begin{array}{cc} 0 & 1 \end{array} \\
 & \begin{pmatrix} 0.44 & 0.55 \end{pmatrix} \\
 & \begin{array}{cc} 0 & 1 \end{array} \\
\begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 0.19 & 0.25 \\ 0.25 & 0.31 \end{pmatrix}
\end{array}
\qquad\qquad
\begin{array}{cc}
 & \begin{array}{cc} 0 & 1 \end{array} \\
 & \begin{pmatrix} 1 & 1 \end{pmatrix} \\
 & \begin{array}{cc} 0 & 1 \end{array} \\
\begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}
\end{array} \tag{22}
$$

These distributions and matrices should be compared to the estimates obtained from the original data set:

$$
\begin{array}{cc}
 & \begin{array}{cc} 0 & 1 \end{array} \\
 & \begin{pmatrix} 0.28 & 0.72 \end{pmatrix} \\
 & \begin{array}{cc} 0 & 1 \end{array} \\
\begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 0.01 & 0.34 \\ 0.22 & 0.44 \end{pmatrix}
\end{array}
\qquad
\begin{array}{cc}
 & \begin{array}{cc} 0 & 1 \end{array} \\
 & \begin{pmatrix} 1 & 1 \end{pmatrix} \\
 & \begin{array}{cc} 0 & 1 \end{array} \\
\begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}
\end{array}
\tag{23}
$$

3 The cycle of rainfed wheat

One 15-term model was obtained for rainfed wheat in the semi-arid climatic region in North Morocco. The model is of maximum degree 3 (q = 3) and reads:

$$
\begin{aligned}
P(X_1, X_2, X_3) = {} & -1.309\,616.10^3 - 4.504\,146.10^3\ X_2 \\
& + 8.897\,279.10^1\ X_2 X_3 - 6.763\,200.10^{-1}\ X_2 X_3^2 - 2.391\,070.10^2\ X_2^2 \\
& - 1.597\,373.10^2\ X_2^3 + 9.416\,550.10^3\ X_1 - 8.571\,862\ X_1 X_3 \\
& + 2.064\,775.10^4\ X_1 X_2 - 1.926\,770.10^2\ X_1 X_2 X_3 \\
& + 4.843\,679.10^2\ X_1 X_2^2 - 2.163\,986.10^4\ X_1^2 + 1.824\,987.10^1\ X_1^2 X_3 \\
& - 2.331\,196.10^4\ X_1^2 X_2 + 1.604\,342.10^4\ X_1^3
\end{aligned}
\tag{24}
$$

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